PREPRINT

Minimal Revision and Classical Kripke Models
First Results

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Abstract Dynamic modal logics are modal logics that have statements of the form $[\pi]\psi$. The truth value of such statements, when evaluated in a pointed model $\langle \mathcal{F}, V, w \rangle$, is determined by the truth value that ψ takes in the pointed models $\langle \mathcal{F}', V', w' \rangle$ that stand in a relation $\xrightarrow{\pi}$ to $\langle \mathcal{F}, V, w \rangle$.
This paper introduces new dynamic operators that minimally revise finite classical Kripke models to make almost any satisfiable modal formula ϕ true. To this end, we define the minimal revision relations $\xrightarrow{\dagger\,\phi}$ and $\xrightarrow{\ddagger\,\phi}$, where $\xrightarrow{\dagger\,\phi}$ revises only the valuation function and $\xrightarrow{\ddagger\,\phi}$ also changes the frame.
We show that our language enables us to count the number of accessible worlds and to characterize irreflexive frames. We also demonstrate that any consistent formula can be made true, conditional only on modest seriality and finiteness presumptions.

Keywords: dynamic modal logic, revision operators, Kripke semantics

1 Multi-modal Kripke semantics

Modal propositional logic extends propositional logic with modalities such as 'necessarily', 'knows that', 'believes that', or 'it ought to be the case that'. It is commonplace to interpret modal logics in terms of Kripke semantics—to the extent that the terms 'modal logic' and 'Kripke semantics' are often treated as if they were synonymous.

Definition 1. *Let the language of multi-modal propositional logic $\mathcal{L}^0$ consists of all well-formed formulas (wffs) ϕ composed as follows:*

$$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi,$$

with p an element of the finite set of atomic propositions Prop *and a an element of the set of indices* Ind.

Read $\neg\phi$ as 'not ϕ' and $(\phi \wedge \psi)$ as 'ϕ and ψ'. $\Box_a \phi$ is pronounced 'box-a ϕ'. By way of example, in epistemic logic this latter formula is interpreted as 'agent a knows that ϕ'. Other logical symbols used in this paper are $\top$ ('top', true in every

Preprint submitted on 15 July 2011. The original publication is available at www.springerlink.com.

world), $\bot$ ('bottom', false in every world), $\vee$ (disjunction), $\rightarrow$ (implication), and $\leftrightarrow$ (equivalence). It is easily seen that these symbols can be reduced in terms of $\neg$ and $\wedge$. Similarly, let $\Diamond_a$ ('diamond') stand for $\neg\Box_a\neg$.

Where in propositional logic you might want to know if a wff holds in a model, in Kripke semantics you would want to know if a wff holds at a 'world' in a model. Let us first look at how these models are structured.

Definition 2. *A multi-modal Kripke ≪frame≫ $\mathcal{F}$ is a pair (W, R), with W a nonempty set of point-like 'worlds' and $R : \mathsf{Ind} \rightarrow (W \times W)$ a function from indices to accessibility relations. The domain W of $\mathcal{F}$ is also denoted* $\mathrm{dom}(\mathcal{F})$*. Additionally, a multi-modal ≪pointed frame≫ is a pair $(\mathcal{F}, w)$, with (W, R) a multi-modal Kripke frame and $w \in \mathrm{dom}(\mathcal{F})$. We call w the current world.*

Note that a world resembles a point in that only its unique identity is relevant for our purposes. Whenever $wR(a)v$ we will say that v is accessible from w over a. Also, from here on we will use the notation wR_av instead. The intuitive understanding of the accessibility relation depends on the application. For instance, in epistemic logic wR_av means that world w represents agent a as not being able to tell apart v from w.

To evaluate wffs we need both a frame and a valuation function.

Definition 3. *A multi-modal Kripke ≪model≫ is a pair $(\mathcal{F}, V)$, with $\mathcal{F} = (W, R)$ a multi-modal Kripke frame and $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ a function from atomic propositions to the sets of worlds these propositions hold in. V is called a valuation function for $\mathcal{F}$ and a proposition $p \in \mathsf{Prop}$ is said to be true at (or hold at) $w \in W$ if and only if $w \in V(p)$. Additionally, a multi-modal ≪pointed model≫ is a tuple $\langle \mathcal{F}, V, w\rangle$, with $(\mathcal{F}, V)$ a multi-modal Kripke model and $w \in \mathrm{dom}(\mathcal{F})$.*

We can now have a look at the formal definitions of the formulas in $\mathcal{L}^0$.

Definition 4. *With $p \in \mathsf{Prop}, a \in \mathsf{Ind}$, and $\mathcal{F} = (W, R)$, define the ≪forcing relation≫ $\Vdash$ between pointed multi-modal Kripke models and formulas of $\mathcal{L}^0$ as follows:*[1]

$$
\begin{aligned}
\langle \mathcal{F}, V, w\rangle \Vdash p &\iff w \in V(p)\\
\langle \mathcal{F}, V, w\rangle \Vdash \neg\phi &\iff \textit{not } \langle \mathcal{F}, V, w\rangle \Vdash \phi\\
\langle \mathcal{F}, V, w\rangle \Vdash (\phi \wedge \psi) &\iff \langle \mathcal{F}, V, w\rangle \Vdash \phi \textit{ and } \langle \mathcal{F}, V, w\rangle \Vdash \psi\\
\langle \mathcal{F}, V, w\rangle \Vdash \Box_a\,\phi &\iff \forall x \in R_a[w] : \langle \mathcal{F}, V, x\rangle \Vdash \phi
\end{aligned}
$$

Notice that we evaluate the modal formula $\Box_a\,\phi$ by quantifying over the worlds that are accessible from the current world over a. Specifically, $\Box_a\,\phi$ is defined to be true if and only if ϕ evaluates to true in all the worlds accessible over a. For example, in the terminology of epistemic logic this means that an agent a knows that ϕ if and only if ϕ is true in all the worlds that a cannot distinguish from the current world.

Depending on the application the accessibility relations are restricted in various ways. For instance, in epistemic logic reflexivity is imposed, which has the

[1] For any relation $Q : X \times Y$ let $Q[x] \coloneqq \{y \in Y \mid xQy\}$.

effect of making $(\Box_a \phi \to \phi)$ valid. This corresponds to the notion that it is impossible to know false propositions. Of course it is possible to *believe* false propositions and hence in doxastic logic axiom, in which $\Box_a \phi$ is read as 'agent a believes that ϕ', the requirement that accessibility relations are reflexive is dropped. Instead seriality is substituted for reflexivity so that agents cannot believe contradictions. Restrictions on the accessibility relations are known as frame conditions.

Definition 5. *Given a number of frame conditions σ, we say that $\mathcal{F} = (W, R)$ is a ≪σ-frame≫ if and only if R_a meets the conditions of σ (for all $a \in \mathsf{Ind}$).*[2] *Similarly, a model that has a σ-frame is called a ≪σ-model≫.*

Finally, we define a notion of equivalence on the level of pointed models.

Definition 6. *A ≪bisimulation≫ between (the worlds of) two models $(\mathcal{F}, V)$ and $(\mathcal{F}', V')$ is a relation ρ between W and W' such that $x\rho x'$ if and only if the following conditions are met:*[3]

Atom equivalence. $x \in V(p) \iff x' \in V'(p)$ *for all* $p \in \mathsf{Prop}$.
Forth. *For every* $a \in \mathsf{Ind}$ *and* $y \in R_a[x]$ *there is a* $y' \in R'_a[x']$ *such that* $y\rho y'$.
Back. *For all* $a \in \mathsf{Ind}$ *and* $y' \in R'_a[x']$ *there is a* $y \in R_a[x]$ *such that* $y\rho y'$.

Furthermore, two pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}', V', w'\rangle$ are bisimilar if and only if there is a bisimulation ρ between them such that $w\rho w'$.

2 Two fixed-frame minimal revision operators

In this section we extend the language of multi-modal logic by introducing minimal revision operators that, given a well-formed formula ϕ to make true, quantify over alternative valuation functions in which ϕ holds. As we will see, if there is a valuation function that makes ϕ true at the current world then the minimal revision operators quantify over at least one such function. Moreover, if there is more than one such function then these operators only quantify over the ones that are only minimally different from the original valuation function.

Let us first look at the grammar of our fixed-frame minimal revision logic.

Definition 7. *With $p \in \mathsf{Prop}$ and $a \in \mathsf{Ind}$, let $\mathcal{L}^\dagger$ be the set of wffs*[4] *ϕ of the following form:*

$$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid [\dagger\, \phi]\, \phi.$$

Read $[\dagger\, \phi]\, \psi$ as 'ψ must hold when ϕ is made true'. Additionally, define $\langle \dagger\, \phi\rangle\, \psi$ as $\neg\, [\dagger\, \phi]\, \neg\psi$ and read it as 'ψ could hold after making ϕ true'.

Next, we make the forcing relation accommodate the $[\dagger\, \phi]$ operators.

[2] No particular formal representation of frame conditions is presupposed in this paper.
[3] Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$.
[4] In this section the term wff is used to denote the elements of $\mathcal{L}^\dagger$.

Definition 8. *Let the forcing relation* $\Vdash$ *be as before, except that* $\mathcal{L}^\dagger$ *is the new codomain and except for the addition of the following rule.*

$$\langle \mathcal{F}, V, w\rangle \Vdash [\dagger\,\phi]\,\psi \iff \forall V^\star : \textit{if } V^\star \textit{ is a valuation function for } \mathcal{F} \textit{ and } \langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle \textit{ then } \langle \mathcal{F}, V^\star, w\rangle \Vdash \psi.$$

It should be clear that we only need to define $\xrightarrow{\dagger\,\phi}$ between pointed models that share the same frame and current world. Moreover, we want $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ to hold if and only if (i) $\langle \mathcal{F}, V^\star, w\rangle \Vdash \phi$ and (ii) $V^\star$ is only 'minimally different' from V. To help us with the second requirement we first define a function δ that, given two valuation functions V and V' for a single frame, tells us which pairs of worlds and atomic propositions are assigned a different truth value by V and V'.

Definition 9. *For any two models* $(\mathcal{F}, V)$ *and* $(\mathcal{F}, V')$*, let*

$$\delta(\mathcal{F}, V, V') \coloneqq \{(w, p) \in W \times \mathsf{Prop} \mid V(p) \cap \{w\} \neq V'(p) \cap \{w\}\}\,.$$

In other words, $\delta(\mathcal{F}, V, V')$ yields a set of pairs (w, p) such that $w \in V(p)$ but $w \notin V'(p)$ or (vice versa) $w \notin V(p)$ but $w \in V'(p)$.

We can now define $\xrightarrow{\dagger\,\phi}$.

Definition 10. *For all* $\phi \in \mathcal{L}^\dagger$ *and pointed models* $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}, V^\star, w\rangle$*, let* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ *if and only if*

1. $\langle \mathcal{F}, V^\star, w\rangle \Vdash \phi$ *and*
2. *There is no valuation* V' *for* $\mathcal{F}$ *such that*
   (a) $\langle \mathcal{F}, V', w\rangle \Vdash \phi$ *and*
   (b) $\delta(\mathcal{F}, V, V') \subset \delta(\mathcal{F}, V, V^\star)$.

One property that can be read off the definition of the fixed-frame minimal revision relation straight away is that $\xrightarrow{\dagger\,\phi}$ only holds between two pointed models if ϕ is true in the right hand model.

Proposition 1 (Success). $\vDash [\dagger\,\phi]\,\phi$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}, V^\star, w\rangle$ *it is the case that* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ *implies that* $\langle \mathcal{F}, V^\star, w\rangle \Vdash \phi$.

This success is not a vacuous accomplishment. If there is a way to revise a *finite* model to make ϕ true (while keeping the frame and current world constant) then there is a minimal revision that makes ϕ true. In other words, the minimal revision operators preserve consistency for finite frames.

Proposition 2 (Finite Consistency). $(\mathcal{F}, w) \vDash [\dagger\,\phi]\,\bot \implies (\mathcal{F}, w) \vDash \neg\phi$
For all finite pointed models $\langle \mathcal{F}, V, w\rangle$*, if a wff* ϕ *is satisfiable at* $(\mathcal{F}, w)$ *then there is a pointed model* $\langle \mathcal{F}, V^\star, w\rangle$ *such that* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$.

Table 1. The AGM postulates for belief revision. In this table K is a belief set, Cn closes a set of formulas under entailment, $\star$ is a belief revision operation, and $+$ is a belief expansion operation. See also [6]

Closure	$K \star \phi = \mathrm{Cn}(K \star \phi)$
Success	$\phi \in K \star \phi$
Inclusion	$K \star \phi \subseteq K + \phi$
Vacuity	$\neg\phi \notin K \implies K \star \phi = K + \phi$
Consistency	$K \star \phi$ is consistent if ϕ is consistent
Extensionality	$(\phi \leftrightarrow \psi) \in \mathrm{Cn}(\emptyset) \implies K \star \phi = K \star \psi$
Superexpansion	$K \star (\phi \wedge \psi) \subseteq (K \star \phi) + \psi$
Subexpansion	$\neg\psi \notin \mathrm{Cn}(K \star \phi) \implies (K \star \phi) + \psi \subseteq K \star (\phi \wedge \psi)$

Proof. Consider first the following set S.

$$S = \{\delta(\mathcal{F}, V, V') \mid V' \text{ is a valuation function for } \mathcal{F} \text{ and } \langle \mathcal{F}, V', w\rangle \Vdash \phi\}$$

We need to show that if S is nonempty then S has minimal elements. To this end we prove that S is finite.

Observe that by definition of δ it is the case that $\delta(\mathcal{F}, V, V') \subseteq \mathrm{dom}(\mathcal{F}) \times \mathsf{Prop}$ for all valuation functions V' for $\mathcal{F}$. And as by stipulation $\mathrm{dom}(\mathcal{F})$ and Prop are finite sets so is their Cartesian product $T = \mathrm{dom}(\mathcal{F}) \times \mathsf{Prop}$. And then so is $\mathcal{P}(T)$. It follows that S is finite since $S \subseteq \mathcal{P}(T)$.

A reader familiar with the belief revision literature may recognize the previous two propositions as AGM postulates for belief revision (Table 1). This is also the origin of the following two propositions. The first one of which is a trivial result from the fact that we only take semantic elements into account when revising models.

Proposition 3 (Extensionality). $\vDash(\phi \leftrightarrow \psi) \implies \vDash([\dagger\, \phi]\, \chi \leftrightarrow [\dagger\, \psi]\, \chi)$
If $\vDash(\phi \leftrightarrow \psi)$ *then for all pointed models* $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}, V^\star, w\rangle$ *it is the case that* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ *if and only if* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\psi} \langle \mathcal{F}, V^\star, w\rangle$.

The minimal revision operators discussed in this paper do not have analogues for all AGM postulates because they revise pointed models and not belief sets. Specifically, they do not have analogues for the AGM postulates that relate belief revision to belief expansion. Vacuity is one such postulate, although we can record the following proposition about the case where ϕ is to be made true in a pointed model in which ϕ is already true (rather than 'not disbelieved').

Proposition 4 (Special Vacuity). $\vDash((\phi \wedge \psi) \rightarrow [\dagger\, \phi]\, \psi)$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}, V^\star, w\rangle$ *such that* $\langle \mathcal{F}, V, w\rangle \Vdash \phi$ *and* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ *it is the case that* $V = V^\star$.

As is typical for dynamic modal operators, $[\dagger\, \phi]$ is 'almost' a normal modal operator. Specifically, we have the Rule of Necessitation and the K Axiom, but not the Substitution Property.

Proposition 5 (Rule of Necessitation). $\vDash \psi \implies \vDash [\dagger\,\phi]\,\psi$

Proof. Whenever $[\dagger\,\phi]\,\psi$ is evaluated in a pointed model $\langle \mathcal{F}, V, w\rangle$, the operator $[\dagger\,\phi]$ quantifies only over models that have the frame $\mathcal{F}$. Hence everything that is valid in $\mathcal{F}$ is the case in all revised models.

Proposition 6 (K). $\vDash([\dagger\,\phi](\chi \to \xi) \to ([\dagger\,\phi]\,\chi \to [\dagger\,\phi]\,\xi))$

Proposition 7. $\vDash \psi \not\Longrightarrow \vDash \psi[p/\phi]$

Proof. It can easily be seen that $\vDash((p \wedge q) \to [\dagger\,\neg\mathrm{p}]\,q)$ but that $\nvDash((p \wedge p) \to [\dagger\,\neg\mathrm{p}]\,p)$.

We now take a look at the relational properties of $\xrightarrow{\dagger\,\phi}$ over the set of all pointed models $\langle \mathcal{F}, V, w\rangle$ that share the same frame $\mathcal{F}$ and current world $w \in \mathrm{dom}(\mathcal{F})$. From this vantage point we get two important properties.

Proposition 8 (Shift Reflexive). $\vDash [\dagger\,\phi]([\dagger\,\phi]\,\psi \to \psi)$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V^\star, w\rangle$ it is the case that if $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ then $\langle \mathcal{F}, V^\star, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$.

Proof. The premise $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ implies that $\langle \mathcal{F}, V^\star, w\rangle \Vdash \phi$. Additionally, it is self-evident that $\delta(\mathcal{F}, V^\star, V^\star) = \emptyset$. Hence there cannot be a pointed model $\langle \mathcal{F}, V', w\rangle$ such that $V' \neq V^\star$ and such that $\langle \mathcal{F}, V', w\rangle \Vdash \phi$ and $\delta(\mathcal{F}, V^\star, V') \subset \delta(\mathcal{F}, V^\star, V^\star)$. It follows that $\langle \mathcal{F}, V^\star, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$.

Proposition 9 (Antisymmetric).
For all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V', w\rangle$, if $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V', w\rangle$ and $\langle \mathcal{F}, V', w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V, w\rangle$ then $V = V'$.

Proof. From $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V', w\rangle$ it follows that $\langle \mathcal{F}, V', w\rangle \Vdash \phi$. Furthermore, from $\langle \mathcal{F}, V', w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V, w\rangle$ it follows that there is no pointed model $\langle \mathcal{F}, V'', w\rangle$ such that $\langle \mathcal{F}, V'', w\rangle \Vdash \phi$ and such that $\delta(\mathcal{F}, V', V'') \subset \delta(\mathcal{F}, V', V)$. Since it is self-evident that $\delta(\mathcal{F}, V', V') = \emptyset$ and since we already established that $\langle \mathcal{F}, V', w\rangle \Vdash \phi$, however, this means that $\delta(\mathcal{F}, V', V) = \delta(\mathcal{F}, V', V') = \emptyset$. It follows that $V = V'$.

Propositions 8 and 9 can be summarized in the following property.

Proposition 10 (Shift Unique and Shift Reflexive). $\vDash [\dagger\,\phi]([\dagger\,\phi]\,\psi \leftrightarrow \psi)$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V^\star, w\rangle$ it is the case that if $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ then $\xrightarrow{\dagger\,\phi} [\langle \mathcal{F}, V^\star, w\rangle] = \{\langle \mathcal{F}, V^\star, w\rangle\}$.

It is now also easy to see that $\xrightarrow{\dagger\,\phi}$ is transitive and dense.[5] Moreover, on the basis of their relational properties with respect to $\xrightarrow{\dagger\,\phi}$ we can partition all pointed models as follows:

[5] A relation $Q : S \times S$ is dense if and only if $\forall x, y \in S : xRy \implies \exists z \in S : xRz$ and zRy.

1. Pointed models in which ϕ does not hold and cannot be made true. These models have no incoming or outgoing $\xrightarrow{\dagger\phi}$-links.
2. Pointed models in which ϕ does not hold but for which ϕ can be made true. These models have outgoing $\xrightarrow{\dagger\phi}$-links to pointed models of the third kind and only to such models. They have no incoming $\xrightarrow{\dagger\phi}$-links.
3. Pointed models in which ϕ holds. These models have reflexive links and no other outgoing links.

The addition of the fixed-frame minimal revision operators makes our logic more expressive than $\mathcal{L}^0$. For instance, it is possible to count the number of accessible worlds over an index.

Theorem 1. *Given a pointed model $\langle \mathcal{F}, V, w\rangle$ it is the case that at least n worlds are accessible from w over a if and only if*

$$\langle \mathcal{F}, V, w\rangle \Vdash \langle \dagger(\Diamond_{\mathrm{a}}\, \phi_1 \wedge \cdots \wedge \Diamond_{\mathrm{a}}\, \phi_{\mathrm{n}})\rangle \top,^{6}$$

where $\phi_1, \ldots, \phi_n$ are satisfiable conjunctions of literals such that no conjunction $(\phi_i \wedge \phi_j)$ is satisfiable (with $i \neq j$).

Proof. Let $\alpha := (\Diamond_a\, \phi_1 \wedge \cdots \wedge \Diamond_a\, \phi_n)$. By definition, $\langle \mathcal{F}, V, w\rangle \Vdash \langle \dagger\, \alpha\rangle \top$ if and only if there is a valuation $V^\star$ for $(\mathcal{F}, w)$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\alpha} \langle \mathcal{F}, V^\star, w\rangle$. By Propositions 1–2 such a valuation function exists if and only if α is satisfiable at $(\mathcal{F}, w)$. Finally, since every ϕ_i is satisfiable but no two ϕ_j and ϕ_k can be true at a single world, α is satisfiable in $(\mathcal{F}, w)$ if and only if there are at least n worlds accessible from w over a.

In $\mathcal{L}^\dagger$ it is also possible to characterize certain types of frames that famously cannot be characterized in $\mathcal{L}^0$. For instance, the following proposition explains how to characterize irreflexive frames (when $n = 0$ and $m = 1$).

Theorem 2. *For all pointed models $\langle \mathcal{F}, V, w\rangle$ and atoms $p \in \mathsf{Prop}$ it is the case that worlds accessible from w over a in n steps are not accessible over a in m steps if and only if $\langle \mathcal{F}, V, w\rangle \Vdash [\dagger\, \Box^{\mathrm{n}}_{\mathrm{a}}\, \mathrm{p}]\, [\dagger\, \Box^{\mathrm{m}}_{\mathrm{a}}\, \neg \mathrm{p}]\, \Box^n_a\, p$.*[7]

Proof. We prove the left to right part of this proposition by contradiction. Suppose that all worlds accessible from w over a in n steps were indeed not accessible over a in m steps. Additionally, suppose that it was not the case that $\langle \mathcal{F}, V, w\rangle \Vdash [\dagger\, \Box^{\mathrm{n}}_{\mathrm{a}}\, \mathrm{p}]\, [\dagger\, \Box^{\mathrm{m}}_{\mathrm{a}}\, \neg \mathrm{p}]\, \Box^n_a\, p$.

For all valuation functions V' for $\mathcal{F}$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\, \Box^n_a\, p} \langle \mathcal{F}, V', w\rangle$ it is the case that $\langle \mathcal{F}, V', w\rangle \Vdash \Box^n_a\, p$ by Proposition 1. That is, p holds in all worlds accessible over a in n steps. Of course for all valuation functions V'' for $\mathcal{F}$ such that $\langle \mathcal{F}, V', w\rangle \xrightarrow{\dagger\, \Box^m_a\, \neg p} \langle \mathcal{F}, V'', w\rangle$ it is the case that p does *not* hold in any of the

[6] Note that for this to be an actual wff extra parentheses would have to be added.

[7] Here $\Box^n_a\, \phi$ stands for $\underbrace{\Box_a \ldots \Box_a}_{n}\, \phi$.

worlds accessible over a in m steps. It is a premise of the left to right part of this proof that p also fails to hold in one of the worlds accessible from w in n steps. Call one such world v. Now either v is accessible from w in m steps and then we have arrived at a contradiction (thereby concluding the first part of the proof). Suppose that, on the other hand, v is not accessible from w in m steps. Consider then the valuation function V''' such that $\delta(\mathcal{F}, V'', V''') = \{(v, p)\}$. Since v would not be accessible in m steps it would still hold that $\langle \mathcal{F}, V''', w \rangle \Vdash \Box_a^m \neg p$. Note however that $\delta(\mathcal{F}, V', V''') = \delta(\mathcal{F}, V', V'') \setminus \{(v, p)\}$. Our previously established fact that $\langle \mathcal{F}, V', w \rangle \xrightarrow{\dagger \Box_a^m \neg p} \langle \mathcal{F}, V'', w \rangle$ is contradicted by the conclusions that (i) $\langle \mathcal{F}, V''', w \rangle \Vdash \phi$ and (ii) $\delta(\mathcal{F}, V', V''') \subset \delta(\mathcal{F}, V', V'')$. So again we found a contradiction that concludes the left to right part of this proof.

We also prove the right to left part of this proposition by contradiction. Suppose that it was indeed the case that $\langle \mathcal{F}, V, w \rangle \Vdash [\dagger \Box_{\mathrm{a}}^{\mathrm{n}} \mathrm{p}] [\dagger \Box_{\mathrm{a}}^{\mathrm{m}} \neg \mathrm{p}] \Box_a^n p$. Suppose also that there was a world v accessible from w over a in n steps that was also accessible over a in m steps. For all valuation functions V' for $\mathcal{F}$ such that $\langle \mathcal{F}, V, w \rangle \xrightarrow{\dagger \Box_a^n p} \langle \mathcal{F}, V', w \rangle$ it would then be the case that $\langle \mathcal{F}, V', v \rangle \Vdash p$. However, for all valuation functions V'' for $\mathcal{F}$ such that $\langle \mathcal{F}, V', w \rangle \xrightarrow{\dagger \phi} \langle \mathcal{F}, V'', w \rangle$ it would be the case that $\langle \mathcal{F}, V'', v \rangle \Vdash \neg p$. But then it would not be the case that $\langle \mathcal{F}, V'', w \rangle \Vdash \Box_a^n p$, which contradicts the premise of the right to left part of this proof.

3 Two generalized minimal revision operators

The fixed-frame operators defined in the previous section allow us to make many formulas true. They are constrained by the available frame, however. Building upon the fixed-frame relation $\xrightarrow{\dagger \phi}$ we will now define a more general relation $\xrightarrow{\ddagger \phi}$ between pointed models. But first we extend our language with generalized minimal revision operators.

Definition 11. *With* $p \in \mathsf{Prop}$ *and* $a \in \mathsf{Ind}$*, let* $\mathcal{L}^\ddagger$ *be the set of wffs* ϕ *of the following form:*

$$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid [\dagger \phi] \phi \mid [\ddagger \phi] \phi.$$

Additionally, let $\langle \ddagger \phi \rangle$ be a shorthand for $\neg [\ddagger \neg\phi]$.

As before we extend the forcing relation. This time around, though, we also add the frame conditions as a parameter.

Definition 12. *Given frame conditions* σ*, let the forcing relation* $\Vdash_\sigma$ *be as* $\Vdash$*, except that* $\mathcal{L}^\ddagger$ *is the new codomain and except for the addition of the following rule.*

$$\langle \mathcal{F}, V, w \rangle \Vdash_\sigma [\ddagger \phi] \psi \iff \forall \langle \mathcal{F}^\star, V^\star, w^\star \rangle : \textit{if } \langle \mathcal{F}^\star, V^\star, w^\star \rangle \textit{ is a pointed } \sigma\textit{-model and } \langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger \phi} \langle \mathcal{F}^\star, V^\star, w^\star \rangle \textit{ then } \langle \mathcal{F}^\star, V^\star, w^\star \rangle \Vdash_\sigma \psi.$$

The generalized minimal revision relation $\xrightarrow{\ddagger\,\phi}$ is defined as follows.

Definition 13. *For all formulas $\phi \in \mathcal{L}^{\ddagger}$ and all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$, let $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ if and only if there is a valuation V' for $\mathcal{F}^\star$ such that*

1. *$\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}^\star, V', w^\star\rangle$ and*
2. *$\langle \mathcal{F}^\star, V', w^\star\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.*

Let us now go through some of the formal properties of these generalizations.

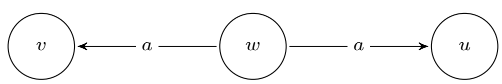
First of all, it turns out that the fixed-frame minimal revision relation is a subset of the generalized minimal revision relation.

Proposition 11. $\vDash_\sigma (\langle \dagger\,\phi\rangle\,\psi \rightarrow \langle \ddagger\,\phi\rangle\,\psi)$ *and* $\vDash_\sigma ([\ddagger\,\phi]\,\psi \rightarrow [\dagger\,\phi]\,\psi)$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V^\star, w\rangle$ it is the case that if $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^*, w\rangle$ then $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}, V^*, w\rangle$.

Note that this property does not hold from right to left, even when keeping the frame and current world constant.

Proposition 12. *It is not always the case that for pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V^\star, w\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$ also $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$.*

Proof. Consider the following frame $\mathcal{F}$.



Consider also two valuation functions V, V' for $\mathcal{F}$ that differ only in their valuation for p at v and u: Let $V(p) = \{v\}$ and $V'(p) = \{u\}$. Since in either case there is one p-world and one $\neg p$-world accessible from w (and apart from those worlds' identities there are no differences), $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}, V', w\rangle$. However, whereas $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\top} \langle \mathcal{F}, V, w\rangle$ it is not so that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\top} \langle \mathcal{F}, V', w\rangle$ since $\delta(\mathcal{F}, V, V) = \emptyset$ but $\delta(\mathcal{F}, V, V') = \{(v, p), (u, p)\}$.

On the other hand, we do have $\langle \mathcal{F}, V', w\rangle \xrightarrow{\dagger\,\top} \langle \mathcal{F}, V', w\rangle$ and $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}, V', w\rangle$ and this entails that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\top} \langle \mathcal{F}, V', w\rangle$.

Additionally, because for all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ there is a valuation function V' for $\mathcal{F}^\star$ such that $\langle \mathcal{F}^\star, V', w^\star\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$, many properties of $\xrightarrow{\dagger\,\phi}$ are inherited by $\xrightarrow{\ddagger\,\phi}$.

Proposition 13 (Success). $\vDash_\sigma [\ddagger\,\phi]\,\phi$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ it is the case that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ implies that $\langle \mathcal{F}^\star, V^\star, w^\star\rangle \Vdash_\sigma \phi$.

Like its fixed-frame variant, the generalized minimal revision relation is consistent and extensional.

Proposition 14 (Finite Consistency). *For all pointed models $\langle \mathcal{F}, V, w\rangle$ it is the case that if there are finite pointed models $\langle \mathcal{F}', V', w'\rangle$ and $\langle \mathcal{F}', V'', w'\rangle$ such that $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w'\rangle$ and $\langle \mathcal{F}', V'', w'\rangle \Vdash \phi$, then there is a pointed model $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.*

Theorem 3 (Finite D-Consistency). $\vDash_\sigma [\ddagger\phi]\bot \implies \vDash_\sigma \neg\phi$
If σ includes seriality and if $\phi \in \mathcal{L}^\ddagger$ is satisfiable in a finite σ-frame then for all finite pointed σ-models $\langle \mathcal{F}, V, w\rangle$ there is a pointed model $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

Proof. As $\langle \mathcal{F}, V, w\rangle$ is a finite σ-model and as ϕ is satisfiable in a finite σ-frame, a finite pointed σ-model $\langle \mathcal{F}^\circ, V^\circ, w^\circ\rangle$ can be constructed such that $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}^\circ, V^\circ, w^\circ\rangle$ and such that ϕ is satisfiable at $(\mathcal{F}^\circ, w^\circ)$. But first, let $(\mathcal{F}', w')$ be a finite frame such that ϕ is satisfiable at $(\mathcal{F}', w')$. Where $\mathcal{F} = (W, R)$, $\mathcal{F}' = (W', R')$, and $\mathcal{F}^\circ = (W^\circ, R^\circ)$, construct $\langle \mathcal{F}^\circ, V^\circ, w^\circ\rangle$ as follows:

- $W^\circ = W \times W'$
- $R^\circ_a : W^\circ \times W^\circ$ such that $(x, x')R^\circ_a(y, y') \iff xR_a y$ and $x'R'_a y'$
- $V^\circ : \mathsf{Prop} \to \mathcal{P}(W^\circ)$ such that $V^\circ(p) \coloneqq V(p) \times W'$
- $w^\circ = (w, w')$

Notice that the frame is computed as in action model logic. The valuation function, on the other hand, ignores W' and is defined solely in terms of W. We show that one bisimulation ρ between $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}^\circ, V^\circ, w^\circ\rangle$ is as follows:

$$\rho : W \times W^\circ \text{ such that } x\rho(y, y') \iff x = y$$

For consider the following:

- Atom equivalence. For all $x \in W$ and $x' \in W'$ such that $x\rho(x, x')$, by definition $x \in V(p) \iff (x, x') \in V^\circ(p)$ for all $p \in \mathsf{Prop}$.
- Forth. We need to prove that for all $x \in W, x' \in W'$, if $x\rho(x, x')$ then for every $a \in \mathsf{Ind}$ and $y \in R_a[x]$ there is a $(y, y') \in R^\circ_a[(x, x')]$ such that $y\rho(y, y')$. By definition of R°_a, however, $(y, y') \in R^\circ_a[(x, x')]$ if and only if (i) $y \in R_a[x]$ and (ii) $y' \in R'_a[x']$. Now (i) is the antecedent of the implication we are interested in so we can assume it holds (since if it doesn't then we're in the clear also). As for (ii), there is at least one $y' \in R'_a[x']$ because we presuppose that $\mathcal{F}'$ is a serial frame.
- Back. We need to prove that for all $x \in W, x' \in W'$, if $x\rho(x, x')$ then for every $a \in \mathsf{Ind}$ and $(y, y') \in R^\circ_a[(x, x')]$ there is a $y \in R_a[x]$ such that $y\rho(y, y')$. By definition of R°_a, however, $(y, y') \in R^\circ_a[(x, x')]$ if and only if (i) $y \in R_a[x]$ and (ii) $y' \in R'_a[x']$. Thus the antecedent of the implication we are interested in—viz. $(y, y') \in R^\circ_a[(x, x')]$—entails its consequent $y \in R_a[x]$.

Additionally, it is obvious that $w\rho(w, w')$. An entirely analogous argument shows that ϕ is satisfiable in $(\mathcal{F}^\circ, w^\circ)$. We omit this argument here.

Moreover, observe that since W and W' are finite, so is its Cartesian product $W^\circ = W \times W'$. It is also easily seen that since R and R' are serial relations so is R°.

Finally, by Proposition 2 there is a pointed model $\langle \mathcal{F}^\circ, V^\star, w^\circ \rangle$ such that $\langle \mathcal{F}^\circ, V^\circ, w^\circ \rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}^\circ, V^\star, w^\circ \rangle$. Since $\langle \mathcal{F}, V, w \rangle \underline{\leftrightarrow} \langle \mathcal{F}^\circ, V^\circ, w^\circ \rangle$ this means that $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\circ, V^\star, w^\circ \rangle$.

It should be obvious that the generalized minimal revision operators too are extensional.

Proposition 15 (Extensionality). $\vDash_\sigma (\phi \leftrightarrow \psi) \implies \vDash_\sigma ([\dagger\,\phi]\,\chi \leftrightarrow [\dagger\,\psi]\,\chi)$

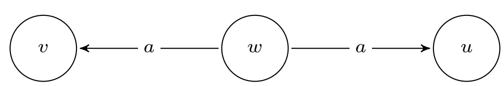
Whereas special vacuity itself does not hold for the generalized minimal revision relation, the following property comes close.

Proposition 16 (Special Quasi-Vacuity). *For all pointed models $\langle \mathcal{F}, V, w \rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star \rangle$ such that $\langle \mathcal{F}, V, w \rangle \Vdash_\sigma \phi$ and $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star \rangle$ it is the case that $\langle \mathcal{F}, V, w \rangle \underline{\leftrightarrow} \langle \mathcal{F}^\star, V^\star, w^\star \rangle$.*

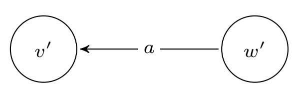
Note, however, that whereas $\vDash((\phi \wedge \psi) \to [\dagger\,\phi]\,\psi)$ corresponds to Proposition 4, the formula $((\phi \wedge \psi) \to [\ddagger\,\phi]\,\psi)$ is not valid for all $\psi \in \mathcal{L}^\ddagger$.

Proposition 17. $\nvDash_\sigma ((\phi \wedge \psi) \to [\ddagger\,\phi]\,\psi)$

Proof. We show that $((\top \wedge \langle \dagger(\Diamond_a \mathrm{p} \wedge \Diamond_a \neg \mathrm{p})\rangle) \wedge \neg [\ddagger\,\top]\,\langle \dagger(\Diamond_a \mathrm{p} \wedge \Diamond_a \neg \mathrm{p})\rangle)$ is satisfiable. First, consider the following frame $\mathcal{F}$:



Now compare $\mathcal{F}$ to the frame $\mathcal{F}$':



Finally, let V and V' be valuation functions for $\mathcal{F}$ and $\mathcal{F}$' such that the extension for every proposition is the empty set. It is obvious that $\langle \mathcal{F}, V, w \rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w' \rangle$. Since, moreover, it is trivially established that $\langle \mathcal{F}', V', w' \rangle \xrightarrow{\dagger\,\top} \langle \mathcal{F}', V', w' \rangle$ we also have $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger\,\top} \langle \mathcal{F}', V', w' \rangle$. However, $\langle \mathcal{F}, V, w \rangle \Vdash_\sigma \langle \dagger(\Diamond_a \mathrm{p} \wedge \Diamond_a \neg \mathrm{p})\rangle\,\top$ but not $\langle \mathcal{F}', V', w' \rangle \Vdash_\sigma \langle \dagger(\Diamond_a \mathrm{p} \wedge \Diamond_a \neg \mathrm{p})\rangle\,\top$.

And again we are dealing with 'almost' normal modal operators.

Proposition 18 (Rule of Necessitation). $\vDash_\sigma \psi \implies \vDash_\sigma [\ddagger\,\phi]\,\psi$

Proof. Trivial result as we restrict quantification for $[\ddagger\,\phi]$ and $\langle\ddagger\,\phi\rangle$ to σ-models.

Proposition 19 (K). $\models_\sigma ([\ddagger\,\phi](\chi \to \xi) \to ([\ddagger\,\phi]\,\chi \to [\ddagger\,\phi]\,\xi))$

Proposition 20. $\models \psi \;\not\Rightarrow\; \models \psi[p/\phi]$

Shift Reflexivity is another property that is preserved.

Proposition 21 (Shift Reflexive). $\models_\sigma [\ddagger\,\phi]([\ddagger\,\phi]\,\psi \to \psi)$
For all pointed models $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$*, if* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ *then* $\langle \mathcal{F}^\star, V^\star, w^\star\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

Antisymmetry and shift uniqueness are *almost* preserved. That is, the derived properties that result from substituting bismilarity for equality hold.

Proposition 22 (Quasi-Antisymmetric). *For all pointed models* $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}', V', w'\rangle$*, if* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}', V', w'\rangle$ *and* $\langle \mathcal{F}', V', w'\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}, V, w\rangle$ *then* $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w'\rangle$.

Proof. From $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}', V', w'\rangle$ it follows that there is a valuation function V° for $\mathcal{F}'$ such that $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V^\circ, w'\rangle$ and such that $\langle \mathcal{F}', V^\circ, w'\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}', V', w'\rangle$. Similarly, from $\langle \mathcal{F}', V', w'\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}, V, w\rangle$ it follows that there is a valuation function $V^\triangleright$ for $\mathcal{F}$ such that $\langle \mathcal{F}', V', w'\rangle \underline{\leftrightarrow} \langle \mathcal{F}, V^\triangleright, w\rangle$ and such that $\langle \mathcal{F}, V^\triangleright, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V, w\rangle$.

From $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}', V', w'\rangle$ it follows that $\langle \mathcal{F}', V', w'\rangle \Vdash_\sigma \phi$ and from $\langle \mathcal{F}', V', w'\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}, V, w\rangle$ it follows that $\langle \mathcal{F}, V, w\rangle \Vdash_\sigma \phi$. But then it also follows that $\langle \mathcal{F}', V^\circ, w'\rangle \Vdash_\sigma \phi$ and that $\langle \mathcal{F}, V^\triangleright, w\rangle \Vdash_\sigma \phi$ since $\langle \mathcal{F}', V^\circ, w'\rangle \underline{\leftrightarrow} \langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}, V^\triangleright, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w'\rangle$.

By Proposition 4 it follows that $V' = V^\circ$ and $V = V^\triangleright$. This, in turn, entails that $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w'\rangle$.

Proposition 23 (Shift Quasi-Unique and Shift Reflexive).
$\models_\sigma [\ddagger\,\phi](\psi \to [\ddagger\,\phi]\,\psi)$
For all finite pointed models $\langle \mathcal{F}, V, w\rangle$ *and* $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ *it is the case that if* $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ *then also* $\{\langle \mathcal{F}^\star, V^\star, w^\star\rangle\} \subseteq \xrightarrow{\ddagger\,\phi} [\langle \mathcal{F}^\star, V^\star, w^\star\rangle] \subseteq \underline{\leftrightarrow}[\langle \mathcal{F}^\star, V^\star, w^\star\rangle]$.

Proof. First. By Proposition 21 it follows from $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ that $\langle \mathcal{F}^\star, V^\star, w^\star\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$. This proves that the set $\xrightarrow{\ddagger\,\phi} [\langle \mathcal{F}^\star, V^\star, w^\star\rangle]$ contains the element $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

Second. Using Proposition 13 we can deduce that $\langle \mathcal{F}^\star, V^\star, w^\star\rangle \Vdash_\sigma \phi$. By Proposition 16 it then follows that for all pointed models $\langle \mathcal{F}', V', w'\rangle$ such that $\langle \mathcal{F}^\star, V^\star, w^\star\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}', V', w'\rangle$ it is the case that $\langle \mathcal{F}', V', w'\rangle \underline{\leftrightarrow} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$. This proves that the set $\xrightarrow{\ddagger\,\phi} [\langle \mathcal{F}^\star, V^\star, w^\star\rangle]$ contains only pointed models that are bisimilar to $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

4 Related Work

Our approach is different from most other dynamic logics in several ways.

First, many dynamic modal logics have operators that directly refer to abstract semantic entities. For instance, in action modal logic—as described in [5]—the dynamic operators refer to 'action models'. New epistemic models are computed by an operation on the current epistemic model and the specified action model. Our dynamic operators instead refer to propositions.

Second, some dynamic modal logics where the dynamic operators refer to propositions only allow a subset of non-modal propositions to be used for this purpose. One example is the logic of 'introspective forgetting' described in [4], which can only express forgetting of atoms. Our operators accept all formulas of the object language.

Third, dynamic modal logics that do permit all well-formed formulas for parameters of dynamic operators typically add extra semantic structure to their models. For instance, in [3] (and many other papers) plausibility orderings are used to sensibly change doxastic models.

We know of two logics that are not different from ours on the above criteria.

First, the dynamic context logic of [2] has dynamic operators for adding and removing ϕ-worlds to and from doxastic models. However, the authors do not offer a criterion for prudently choosing a set of ϕ-worlds to add. Instead they opted to make this a nondeterministic choice. By contrast, in this paper revisions are restricted to *minimal* valuation changes—although we grant that other notions of minimal change exist and that in the absence of applications it is impossible to decide whether our notion of minimality is ultimately a desirable one.

Second, [1] explains how to do AGM belief revision on 'internal belief models' that are sets of pointed multi-modal Kripke models. This is done using centered ordering relations on pointed Kripke models. These orderings are based on a notion of n-bisimilarity, though, and as such only take into account *whether* two worlds have different valuations and ignore the *degree* by which they are different.

From a technical point of view it is interesting that action modal logic and our minimal revision logic have almost the same constraints when it comes to realizing more or less arbitrary formulas. To wit, [5] contains a result that if ϕ is satisfiable in a pointed frame $(\mathcal{F}, w)$ such that its current world is part of a serial subframe $(\mathcal{F}', w)$ then there is an update that realizes ϕ (see Definition 3.1 and Corollary 3.4). This is reminiscent of Theorem 3 in this paper. As a matter of fact, in this respect action modal logic updates are slightly more powerful than minimal revisions as for a minimal revision to be successful it must have a serial subframe $(\mathcal{F}', w)$ that can satisfy a state bisimilar to the original pointed model. Spelling out this property would yield a statement halfway between Proposition 14 and Theorem 3.

5 Future Work

The search for theorems has only just begun and it is not yet clear if the minimal revision operators are axiomatizable. Moreover, many questions remain regarding the expressivity of $\mathcal{L}^{\dagger}$, $\mathcal{L}^{\ddagger}$, and the language that results when the formulas containing fixed-frame operators are subtracted from $\mathcal{L}^{\ddagger}$.

So far we do not have applications for minimal revision. We do, however, intend to investigate whether minimal revision has applications in doxastic and epistemic logic. For instance, we are eager to find out if minimal revision can be used to extend the forgetting operator of [4] so that the forgetting of arbitrary modal formulas can be modeled. It might also be worthwhile to search for 'expansion' operators that validate counterparts of the remaining AGM postulates—i.e. those postulates that relate belief revision to belief expansion.

Finally, we want to explore what happens when the generalized revision relation is decoupled from the fixed-frame relation. For instance, the latter could be made a parameter of the generalized relation. Alternatively, an operator could be introduced that quantifies over the pointed models bisimilar to the current one.

6 Acknowledgements

I thank Patrick Allo for our regular meetings during the genesis of this paper. I also thank Benedikt Löwe as well as Davide Grossi and Peter van Ormondt for previous discussions on closely related topics. Lastly, this paper benefited from the comments by the anonymous reviewers.

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